Calculus Memory Book

for use with

AP CALCULUS BC

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Calculus Memory Book

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PRECALCULUS TOPICS

- $a^2 - b^2 = (a+b)(a-b)$
- Rule of thumb --- Multiply out numerators but keep denominators factored
- Flip the inequality sign over if $\times$ or $\div$ by a negative
- $a^3 + b^3 = (a+b)(a^2-ab+b^2)$
- $a^3 - b^3 = (a-b)(a^2+ab+b^2)$
- $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$
- If you choose to approximate an answer with a decimal, round to **<u>3</u>** places but **not** till the very end of the problem.
- Quadratic Formula: Given $ax^2+bx+c=0$ then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- To solve any *<u>in</u>*equality, mark undefined points and solutions to the *<u>eq</u>*uality on a number line (split points) and test #'s inbetween the split points in the *<u>in</u>*equality (test points).
- $| x | \Rightarrow x$ if $x \geq 0$
  $\Rightarrow -x$ if $x < 0$
- One way to solve absolute value problems is to split them into separate problems without absolute value
- Distance formula $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
- Equation of a Circle with center (h,k) and radius r is $(x-h)^2+(y-k)^2=r^2$
- Midpoint formula: $\left((x_1+x_2)/2 \ , \ (y_1+y_2)/2 \right)$
- Slope of a line: $m = (y_2-y_1)/(x_2-x_1)$
- Point slope formula : $y - y_1 = m(x - x_1)$ (equation of a line)
- $x=a$, vertical line (infinite or undefined slope)
- $y=b$, horizontal line (zero slope)
- Distance from (x_1,y_1) from $Ax+By+C=0$ is $\dfrac{| Ax_1 + By_1 + C |}{\sqrt{A^2 + B^2}}$
- ↕↕ Parallel lines have same slope
- ↕↔ Slopes of perpendicular lines are negative reciprocals
- y-intercept set x=0
- x-intercept set y=0

The graphs of the following functions should be memorized :

- y=mx+b

- y= x^2
- y= x^3
- y= $\sqrt{x}$
- y=1/x
- y= $\sqrt[3]{x}$
- y=lnx
- y=e^x
- y=sinx
- y=cosx
- y=tanx
- You need to memorize the unit circle
- You need to memorize the following identities:

tan x = $\frac{\sin x}{\cos x}$ cot x = $\frac{\cos x}{\sin x}$ sec x = 1 / cos x csc x = 1 / sin x

$\sin^2 x = 1 - \cos^2 x$ $\cos^2 x = 1 - \sin^2 x$

$\sec^2 x = 1 + \tan^2 x$ $\tan^2 x = \sec^2 x - 1$

2sinx cosx = sin2x $\cos^2 x - \sin^2 x = \cos 2x$

$\cos^2 x = \frac{1}{2}(1+\cos 2x)$ $\sin^2 x = \frac{1}{2}(1-\cos 2x)$

sin(-x) = -sinx cos(-x) = cosx tan(-x) = -tan(x)

- Function:
    - For each x-value, there can be only one y-value
    - Domain = all possible values for x
    - Range = all possible values for y
- Even Function : show f(-x) = f(x) (y-axis symmetry)
- Odd function : show f(-x) = - f(x) (origin symmetry)
- to find out where two functions intersect, set them equal & solve for x
- Greatest Integer Function [x] or [[x]] rounds x down to the nearest integer less than or equal to x
- Composite: $(f \circ g)(x)=f(g(x))$
- In calculus we change all logarithms to base e (the natural logarithm)

using: $$\log_a x = \frac{\ln x}{\ln a}$$

- Laws of logarithms

ln(ab)=lna + lnb ln(a/b)=lna - lnb ln(1/a)= - lna $\ln(a^n)=n\ln a$

LIMITS

- The limit is the y-value you are getting close to **not** necessarily the function value itself.
- **Definition of Limit**:

Given any $\in$ > 0 if there is a corresponding number, δ > 0 such that $0 < |x-a| < \delta$ implies $|f(x)-L| < \in$ then $\lim_{x \to a} f(x)=L$

- $\lim_{x \to 0} (\sin x)/x=1$
- $\lim_{x \to 0} (1-\cos x)/x=0$
- $\lim_{x \to \infty} (1 + c/x)^x = e^c$
- $\lim_{x \to 0^+} (1 + cx)^{1/x} = e^c$
- compound interest: $A(t) = A_0 (1 + r/n)^{nt}$
  n = number of times per year compounding takes place   t = # of years
  r = annual interest rate and initial investment = A_0
- continuously compounded = $A_0 e^{rt}$
- As long as each part of a limit problem is defined as a real #, it's ok to evaluate limits piecemeal. (i.e. no neg's under $\sqrt{\ }$, no 0's in denom.)
- **Squeeze Theorem**:
  If $f(x) \leq g(x) \leq h(x)$ and as $x \to a$ $f(x) \to L$ and $h(x) \to L$ then $g(x) \to L$ also.
- **Definition of Continuity**: as $x \to a$ $\lim f(x)= f(a)$
- **Intermediate Value Theorem**: if f(x) is continuous and p is between f(a) and f(b), then there is at least one x-value c between a and b such that f(c)= p.
- **L'Hopital (or L'Hospital) 's Rule** if g(x) and f(x) both $\to 0$ or both $\to \pm\infty$
  then the limit of $\frac{f(x)}{g(x)}$ equals the limit of $\frac{f'(x)}{g'(x)}$
- **Indeterminate forms:** $\frac{0}{0}, \frac{\infty}{\infty}, \infty-\infty, 0 \cdot \infty, \infty^0, 1^\infty, 0^0$
  must be rewritten as quotients before applying L'Hopital's Rule.

DERIVATIVES

- Slope of the secant line through (a,f(a)) and (b,f(b)) equals Average rate of change in f on [a,b]: [f(b)-f(a)]/(b-a)
- Slope of tangent at x=c
  - $M_{tan}= \lim m_{sec}= \lim [f(c+h)-f(c)]/h$ as $h \to 0$
  - or as $x \to c$, $\lim [f(x)-f(c)]/(x-c)$

- These are all the same idea: Slope of tangent line, instantaneous rate of change, slope of curve, derivative
- derivative of f(x) at x=a

  $$f'(a) = \lim_{x\to a} [f(x)-f(a)]/(x-a)$$

- General formula for a derivative (or slope) of f at <u>any</u> x

  $$f'(x) = \lim_{h\to 0} [f(x+h)-f(x)]/h$$

  $$f'(x) = \lim_{t\to x} [f(x)-f(t)]/(x-t)$$

  $$f'(x) = \lim_{\Delta x\to 0} \Delta y/\Delta x = \lim_{\Delta x\to 0} [f(x+\Delta x)-f(x)]/\Delta x$$

- If $f'(c)$ exists, then f is continuous at x=c
- If f is continuous at x=c and not a sharp corner (cusp or singularity – no local linearity) nor vertical, then $f'(c)$ exists
- If $f'(c)$ does not exist, then f is either discontinuous, has a sharp corner at x=c, or has a vertical tangent line at x=c.
- $d/dx\,(c) = 0$
- $d/dx\,(mx+b) = m$
- $d/dx\,(x^n) = n \ast x^{n-1}$ (variable to a constant power)
- $d/dx\,(k \ast f(x)) = k \ast f'(x)$ (Constant Multiplier Rule)
- $d/dx\,(f+g) = f' + g'$
- Product Rule: $d/dx\,[f \ast g] = f \ast g' + g \ast f'$
- Triple Product Rule: $d/dx\,[f \ast g \ast h] = f \ast g \ast h' + f \ast g' \ast h + f' \ast g \ast h$
- Quotient Rule: $[g \ast f' - g' \ast f] / (g^2) = d/dx\,(f/g)$
- $d/dx\,(\sin x) = \cos x$
- $d/dx\,(\cos x) = -\sin x$
- $d/dx\,(\tan x) = \sec^2 x$
- $d/dx\,(\sec x) = \tan x \sec x$
- $d/dx\,(\csc x) = -\csc x \cot x$
- $d/dx\,(\cot x) = -\csc^2 x$
- $d/dx(e^x) = e^x$
- $d/dx(a^x) = (\ln a)(a^x)$
- $d/dx(e^{f(x)}) = e^{f(x)} f'(x)$
- $d/dx(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$ $\quad d/dx(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$
- $d/dx(\tan^{-1} x) = \frac{1}{1+x^2}$ $\quad d/dx(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$
- $d/dx(\ln x) = d/dx(\ln|x|) = d/dx(\ln(-x)) = 1/x$
- $d/dx(\ln(f(x))) = d/dx(\ln|f(x)|) = \frac{f'(x)}{f(x)}$

- d/dx(|x|)= $\frac{|x|}{x}$
- **Chain Rule** (for composites):
  (Newton Notation) $\frac{d}{dx}[f(g(x))] = f'(g(x)) * \mathbf{g'(x)}$
  ↑
  (tag or baby)
- $\frac{dy}{dx}$ ← y is the dependent variable, ← x is the independent variable
  (derivative of y with respect to x)
- **Chain Rule** $\frac{dy}{dx} = \frac{dy}{du} * \frac{du}{dx}$
  (Liebniz notation)

- $\Delta y/\Delta x$ = slope of secant line, dy/dx = slope of tangent line
- $x(t)$ = position
- $x'(t) = v(t)$ = velocity
- $x''(t) = v'(t) = a(t)$ = acceleration
- particle/body is at rest if $v(t)= 0$
- particle/body moves left/down if $v(t) < 0$
- particle/body moves right/up if $v(t) > 0$
- speed = $|v(t)|$
- if a(t) and v(t) have same sign then particle is speeding up
- if a(t) and v(t) have opposite signs then particle is slowing down
- Normal line is perpendicular to the tangent line at the point of tangency. The slope of the normal line through the point of tangency at x = a is:
  $m= -1/ f'(a)$
- d/dx ($\text{variable}^{\text{variable}}$) = ?
  1. Write y = $\text{variable}^{\text{variable}}$
  2. Take the natural log (ln) of both sides & bring down power
  3. Then use implicit differentiation.
- Inverse Function Theorem
  given (a,b) on f(x)⇒ $(f^{-1})'(b) = 1 / f'(a)$
  ↑ ↑
  y-value in original f(x) x-value in original f(x)
- If f(x) changes proportionally (or directly) with its own y-value then
  $dy/dx= ky$ and $f(x) = A_0e^{kx}$

APPLICATIONS OF DERIVATIVES

- In a related rates problem "t" is the independent variable.
  All other variables must be multiplied by the appropriate tag/baby.
  (dV/dt, dy/dt, dA/dt, etc…)

- In a related rates problem, never substitute values for any variable quantities until after you take the derivative with regards to "t". (Otherwise your derivatives will be zero and answers invalid.)
- Given the tangent line $y = mx+b$ tangent to graph at x

$$\begin{aligned} y(x+\Delta x) &\approx m(x+\Delta x)+b \\ &\approx mx+m\Delta x+b \\ &\approx f(x)+f'(x)*\Delta x \text{ where } f'(x)*\Delta x = dy \approx \Delta y \end{aligned}$$

- $f(x+\Delta x) \approx f(x)+f'(x)*\Delta x$ or $f(x+h) \approx f(x)+f'(x)*h$
  By itself $f'(x)*\Delta x$ is called the "error" or approximate change in f
  The tangent line at (c,f(c)) is: $y = f(c)+f'(c)(x-c)$ note: $x-c = \Delta x$
  So just use the tangent line for linear approximations (linearization)
- Mins and Maxes (extrema) can only occur at *critical points* which are:
  - At an endpoint
  - At stationary points $f'(c)=0$
  - At singular points $f'(c)$ is undefined but f(c) is defined
- Stationary points are when $f'(c) = 0$, these are extrema IF AND ONLY IF f' changes signs at x = c
- Inflection points occur where $f''=0$ (or undef.) **and** f'' changes signs
- Horizontal inflection points occur when $f'= 0$ and $f'' = 0$ and f'' changes signs but f' does not
- Monotonicity Theorem:
  - If $f' > 0$ on an interval then f is increasing in the interval
  - If $f' < 0$ on an interval then f is decreasing in the interval
  - If $f' = 0$ on an interval then f is constant on the interval
- Concavity
  - If $f'' > 0$ then f is concave up on the interval
  - If $f'' < 0$ then f is concave down on the interval
  - f can only change concavity where $f''=0$ or is undefined.
  - f'' must also change signs in order for there to actually be an inflection point
- A local (or relative) Max is a y-value greater than neighboring y-values
- A local (or relative) Min is a y-value less than neighboring y-values
- A global (or absolute) Max is the *highest* y-value of all y values in the range
- A global (or absolute) Min is the *lowest* y-value of all y values
- a closed interval guarantees global extrema
- **First Derivative Test**: (always works) Given $f'(c)= 0$ or undefined (with f(c) defined) then:
  - If f' changes from + to –, f(c) is a maximum
  - If f' changes from – to +, f(c) is a minimum
  - The y-value is the minimum or maximum ***value***
  - The x-value is called the ***critical*** value or number

- **Second Derivative Test**:

  Given f '(c)= 0 then:
  - If f " > 0 then f(c) is a minimum ☺
  - If f " < 0 then f(c) is a maximum ☹
  - If f " = 0 or undefined then the second derivative test fails and you must use first derivative test
- ***Never Forget!!*** The absolute Min or Max *could* occur at an endpoint

- **ECONOMICS:**
- P(x)=R(x)-C(x) = profit (profit=revenue – cost)
- p(x) = price
- x= number of units produced/consumed
- C(x)= fixed cost + cost per item $*$ x
- R(x)= x $*$ price of one item
- *Marginal Revenue* = R'(x) *Marginal* profit = P'(x) *Marginal* cost = C'(x)

- Vertical Asymptote (can only occur if denominator → 0, when x → a)
  - $\lim_{x \to \pm a} f(x) = \pm\infty \Rightarrow$ Vertical Asymptote Verification
- Horizontal Asymptote: y = a (to verify…you must take the limit)
  - $\lim_{x \to \pm\infty} f(x) = a \Rightarrow$ Horizontal asymptote
  - Note : if f(x) is a rational function

    $\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty}$ (ratio of leading terms)
- **<u>Mean Value Theorem</u>** (for derivatives)

Given f(x) continuous on [a,b] and differentiable on (a,b) then
f '(c) = [f(b)-f(a)]/(b-a) for at least one value of "c" between a and b.

- Instantaneous rate of change in f at x=c is f ' (c)
- Average rate of change on interval [a,b] is (f(b)-f(a))/(b-a)
- If f '=g' then f-g=c (or f=g+c)

<u>Newton's Method</u> (to $\approx$ a zero of f(x))

- x_1 = # of your choice near the zero (1st approximation)
- $x_2 = x_1 - f(x_1) / f'(x_1)$ (2nd approximation)
- $x_3 = x_2 - f(x_2) / f'(x_2)$ (3rd approximation)
- continue until answers repeat

INTEGRATION

- $\int x^r \, dx = \frac{1}{r+1} x^{r+1} + c$
- $\int \sin x \, dx = -\cos x + c$
- $\int \cos x \, dx = \sin x + c$
- $\int k f(x) \, dx = k \int f(x) \, dx$
- $\int (f(x) \pm g(x)) \, dx = \int f(x) \, dx \pm \int g(x) \, dx$
- $\int [g(x)]^r g'(x) \, dx = \frac{1}{r+1} [g(x)]^{r+1} + c$
- $\int \sec^2 x \, dx = \tan x + c$
- $\int \sec x \tan x \, dx = \sec x + c$
- $\int -\csc^2 x \, dx = \cot x + c$
- $\int \csc x \cot x \, dx = -\csc x + c$
- $\int f'(g(x)) g'(x) \, dx = f(g(x)) + c$
- $\int e^x dx = e^x + C$
- $\int a^x dx = \frac{a^x}{\ln a} + C$
- $\int \frac{1}{x} dx = \ln |x| + C$
- $\int \frac{1}{ax+b} dx = \frac{1}{a} \ln |ax+b| + C$
- $\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$
- $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$

- $\int \sec x dx = \ln|\sec x + \tan x| + C$
- $\int \tan x dx = \ln|\sec x| + C$ = $-\ln|\cos x| + C$
- $\sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n = n(n+1)/2$
- $\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = n(n+1)(2n+1)/6$
- $\sum_{i=1}^{n} c = nc$

- Riemann Sum – any combination of rectangles whose tops intersect the curve of f(x) and which approximate the area between f(x) and the x-axis, using n rectangles from x=a to x=b
- Right sum – use the rightmost x-value on each interval to compute f(x) = height of rectangle
- Left sum – use the leftmost x-value on each interval to compute f(x)
- Upper sum – use largest y-value on each interval for the heights
- Lower sum – use smallest y-value on each interval for the heights
- If $f(x) > 0$ then upper sums are circumscribed and lower sums are inscribed
- If $f(x) < 0$ then upper sums are inscribed and lower sums are circumscribed
- Midpoint sum – use the x-value in the center of each interval
- $\int_a^a f(x)dx = 0$
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^b f(x)dx =$ lim as $n \to \infty$ (any Riemann Sum with n rectangles)
- i.e. $\sum f(x_i)\Delta x_i$ approaches the integral $\int_a^b f(x)dx$
- $f(x_i)$ = height of each rectangle, Δx = width of each rectangle = $\frac{b-a}{n}$,

  and $x_i = a + i * \frac{b-a}{n}$
- $f(x_i) \Rightarrow f(x)$ and $\Delta x \Rightarrow dx$

- **Trapezoidal Rule** – using trapezoids to approximate area, bases are y-values, heights are Δx and each trapezoid's area = ½ h (b_1+b_2)
  A=$\frac{1}{2}$ Δx [f(x_0) + 2 f(x_1) + 2f(x_2) + …+ 2f(x_{n-1})+f(x_n)]
  (Double all but first and last y-values.

- **Fundamental Theorem of Calculus:**

  $\int_a^b f(x)dx = F(b) - F(a)$ where F(x) is any antiderivative of f(x)

  i.e. F'(x)= f(x)

- Version two of the Fundamental Theorem: d/dx [$\int_a^x f(t)dt$] = f(x)

- If f(x)$\geq$0, you get the area

- If f(x)$\leq$0, you get the negative of the area

- any Lower Sum $\leq$ $\int_a^b f(x)dx \leq$ any Upper Sum

- m(b-a)$\leq \int_a^b f(x)dx \leq$M(b-a) (upper and lower sums with one rectangle)

  ↑ ↑
  min on [a,b] max on [a,b]

- d/dx [$\int_{h(x)}^{g(x)} f(t)dt = f(g(x))g'(x) - f(h(x))h'(x)$

- **Mean Value Theorem for Integrals:**

  $\int_a^b f(t)dt = f(c)[b-a]$ for some c between a and b.

  Note: f(c) is called the ***average*** *value* of f(x) on [a,b].

  $$f(c) = \frac{\int_a^b f(x)dx}{b-a} = \frac{F(b)-F(a)}{b-a}$$

(not to be confused with average rate of change in f(x) = $\frac{f(b)-f(a)}{b-a}$)

- **U-substitution**:

If you see an integral with a composite function and a more or less correct tag/baby, then let u equal the inside function, compute du. Change everything totally into u's and integrate. THIS SHOULD BE FULLY MASTERED and the FIRST THING YOU TRY!!!
Don't forget to change a and b also!

- **integration by parts**: $\int f \cdot g' = fg - \int f' \cdot g$ or $\int udv = uv - \int vdu$

- **integration by partial fractions**:

$$\int \frac{f(x)}{(x-a)(x-b)...(x-c)}cx = \int \frac{A}{x-a} + \frac{B}{x-b} + ... + \frac{C}{x-c}dx$$

repeated powers need extra fractions. (ex $\frac{1}{(x-a)^2} = \frac{A}{x-a} + \frac{B}{(x-a)^2}$)

If the degree of the numerator if greater than or equal to the degree of the denominator, then the fraction is "top heavy) and long division must be applied BEFORE using the technique of partial fractions.

- **integration of trigonometric integrals**:
  odd powers of sine and cosine – save one, convert rest using Pythagorean identities
  even powers of sine and cosine – use ½(1 ± cos2x) repeatedly
- **integration by desperation** – try some creative u-substitutions. Try letting u = entire radical and remove the radical before computing du.

APPLICATIONS OF INTEGRALS

- Area between two curves that intersect at x=a and x=b

$$= \int_a^b |f(x) - g(x)|dx$$

(avoid absolute value by placing higher function first)

- Total Distance Traveled

$$= \int_a^b v(t)dt \quad + \quad -\int_c^d v(t)dt \quad \text{or} \quad \int_a^d |v(t)|dt$$

(where v(t)> 0) total dist. moving right (where v(t)< 0) total dist. moving left (total distance moved)

- Displacement

$$\int_a^b v(t)dt$$

where t=a is starting time

where t=b is ending time

(distance between starting position and ending position)

VOLUME BY DISKS & WASHERS

- Rotate about x-axis

$$\int_a^b \pi[f(x)]^2 dx \qquad \text{(disks)}$$

$$\pi \int_a^b \{[f(x)]^2 - [g(x)]^2\}dx \qquad \text{(washers)}$$

Be sure that you ***square the radii separately*** and then subtract!

- Rotate about y-axis:

$$\int_c^d \pi[f(y)]^2\,dy \qquad \text{(disks)}$$

$$\pi\int_c^d \{[f(y)]^2-[g(y)]^2\}dy \qquad \text{(washers)}$$

- Rotate about y=k

$$\int_a^b \pi[k-f(x)]^2\,dx \qquad \text{(disks)}$$

$$\pi\int_a^b \{[k-f(x)]^2-[k-g(x)]^2\}dx \qquad \text{(washers)}$$

- Rotate about x=k

$$\int_c^d \pi[k-f(y)]^2\,dy \qquad \text{(disks)}$$

$$\pi\int_c^d \{[k-f(y)]^2-[k-g(y)]^2\}dy \qquad \text{(washers)}$$

- Arc length
- $\mathcal{L} = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$ or $\int_c^d \sqrt{[f'(y)]^2+1}\,dy$
- Speed along the curve = the derivative of the above integral i.e. $\sqrt{1+[f'(x)]^2}$ etc.
- Surface Area when curve is rotated …

about x-axis: S $= 2\pi\int_a^b y\sqrt{1+[f'(x)]^2}\,dx$ or $2\pi\int_c^d y\sqrt{1+[f'(y)]^2}\,dy$

about y-axis: S $=2\pi\int_a^b x\sqrt{1+[f'(x)]^2}\,dx$ or $2\pi\int_c^d x\sqrt{1+[f'(y)]^2}\,dy$

- **WORK= the integral of force**

W = $\int_a^b F(x)dx$ For springs F(x)=kx

For pumping a liquid W = $\int_a^b (d*a*h)\,dx$

Where d = 9.8 * mass per unit (metric/meters) or weight per unit (standard/pounds) or density

a = area of cross section of liquid to be moved
h =distance each slab must travel
dx = the instantaneous thickness of each cross section
and a to b is original position of liquid
Work units are Joules or foot-pounds
Force units are Newtons or pounds

- Note: average speed = total distance traveled / total time
- average velocity = displacement / total time

SLOPE FIELDS

- Euler's Method

  Create a chart using the following values: $\Delta x, n, x_n, y_n \approx y_{n-1} + \Delta x \cdot y'(x_{n-1}, y_{n-1})$

X	$\Delta y = y'(previous) \cdot \Delta x$	New y
x_0	------------	y_0
$x_0 + \Delta x$	$y'(x_0, y_0) \cdot \Delta x$	$y_0 + \Delta y$
. . .	. . .	. . . .
x_n	$y'(x_n, y_n) \cdot \Delta x$	y_n

Euler's method follows pieces of approximate tangent lines.
These values will be a bit low if f is concave up and a bit too large if f is concave down

Slope fields show lots of little pieces (line segments) of the tangent lines for many possible curves with various possible initial conditions

- Given a differential equation:
  1. separate the variables: f(y) dy= g(x) dx
  2. integrate both sides (+c only on one side)
  3. solve for c
  4. solve for y

Parametric and Polar Functions

- $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{dy}{dt} \times \frac{dt}{dx}$
- $\frac{d^2y}{dx^2} = \frac{d/dt\left(dy/dx\right)}{dx/dt}$
- $\mathcal{L} = \int_\alpha^\beta \sqrt{\left(dx/dt\right)^2 + \left(dy/dt\right)^2}\ dt$
- $\mathcal{S} = \int_\alpha^\beta 2\pi y \sqrt{\left(dx/dt\right)^2 + \left(dy/dt\right)^2}\ dt$ <u>or</u> $\int_\alpha^\beta 2\pi x \sqrt{\left(dx/dt\right)^2 + \left(dy/dt\right)^2}\ dt$

  ↑about x-axis ↑about y-axis

- Speed = $\sqrt{\left(dx/dt\right)^2 + \left(dy/dt\right)^2}$ (derivative of integral for arc length…duh!)

- x = r cos θ
- y = r sin θ
- $x^2 + y^2 = r^2$
- tan θ = y/x
- $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{dr/d\theta}{dr/d\theta} \frac{\sin\theta + r\cos\theta}{\cos\theta - r\sin\theta}$
- $A = \int_\alpha^\beta \tfrac{1}{2}[f(\theta)]\,d\theta = \int_\alpha^\beta \tfrac{1}{2} r^2\,d\theta$
- $\mathcal{L} = \int_\alpha^\beta \sqrt{r^2 + (dr/d\theta)^2}\,d\theta$

- Vector Valued Functions : F = $x\vec{i} + y\vec{j} + z\vec{k}$ or $\langle x, y, z \rangle$

  Where x, y, and z are functions of t

  - $\vec{F}'(t) = x'(t)\vec{i} + y'(t)\vec{j} + z'(t)\vec{k}$
  - $\int_\alpha^\beta \vec{F}(t)dt = \left[\int_\alpha^\beta x(t)dt\right]\vec{i} + \left[\int_\alpha^\beta y(t)dt\right]\vec{j} + \left[\int_\alpha^\beta z(t)dt\right]\vec{k}$
  - $\lim_{t\to a} \vec{F}(t) = [\lim_{t\to a} x(t)]\vec{i} + [\lim_{t\to a} y(t)]\vec{j} + [\lim_{t\to a} z(t)]\vec{k}$
  - Smooth curve = no discontinuities and no sharp corners, and $\vec{F}(t)$ exists and is <u>non</u> zero $\vec{0}$ for all t
  - Vector valued functions are really the same as parametric curves. They just use different notation.

SEQUENCES AND SERIES

Sequences

Converge to L if $\lim_{n\to\infty} a_n = L$ = a real number (diverges otherwise)

Bounded monotonic sequences converge.

When in doubt look at a_{100} or a_{1000} etc. – but watch for calculator error if you make n too big for it to handle

Geometric Series:

$a + ar + ar^2 + ar^3 + \ldots = \frac{a}{1-r}$ only if $-1 < r < 1$

General Series Rules:

- If a_n does not → 0 then $\sum a_n$ diverges
- If $a_n \to 0$ then $\sum a_n$ ***might*** converge (Has to go to 0 “quickly”)
- Convergent series with all positive terms will still converge to the same # even when terms are rearranged.

- Divergent series' very nature can be altered if terms are rearranged… therefore you may NOT rearrange them safely

Integral Test

- $\int_a^\infty f(x)\,dx$ converges if and only if $\sum a_n$ converges
- $\int_a^\infty f(x)\,dx$ diverges if and only if $\sum a_n$ diverges

p – series

- $\sum \frac{1}{n^p}$ converges for p > 1, diverges for p <1
- Harmonic series (p=1) $\sum \frac{1}{n}$ diverges

Comparison Test (for a_n and $b_n \geq 0$)

- If $a_n \geq b_n$ and $\sum b_n$ diverges then so does $\sum a_n$
- If $a_n \leq b_n$ and $\sum b_n$ converges then so does $\sum a_n$

Limit Comparison Test ($a_n \geq 0$)

If $\lim\limits_{n\to\infty} \frac{a_n}{b_n}$ = $L \in R^+$ (*not* zero) then either

$\sum a_n$ and $\sum b_n$ both diverge *or* both converge

Ratio Test ($a_n \geq 0$)

If $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$ = p then

Root test ($a_n \geq 0$)

If $\lim\limits_{n\to\infty} \sqrt[n]{a_n}$ = p then

- p>1 → $\sum a_n$ diverges
- p <1 → $\sum a_n$ converges
- p = 1 → test fails → try comparison test

Alternating Series Test

If $\lim\limits_{n\to\infty} a_n = 0$, $|a_n|$ decreasing *and* $\sum a_n$ alternates then $\sum a_n$ converges

If $\sum a_n$ converges but $\sum |a_n|$ diverges then the convergence of $\sum a_n$ is "<u>conditional</u>"

If both $\sum a_n$ and $\sum |a_n|$ converge then the convergence of $\sum a_n$ is "<u>absolute</u>"

Power Series

$\sum C_n x^n$ will either:

- converge only for x = 0
- converge on [(-*R*,*R*)] for some real number R
- converge for all x

(find this interval of convergence by applying the ratio or root test to $|C_n x^n|$)

Maclaurin Series f(x) ≈ $\sum_{n=0}^{\infty}\frac{f^n(0)}{n!}x^n$

Taylor Series f(x) ≈ $\sum_{n=0}^{\infty}\frac{f^n(a)}{n!}(x-a)^n$

These approximations are only valid if the remainder $r_n(x)$ approaches zero.

La Grange Remainder Theorem

F(x) = $\sum_{k=0}^{n}\frac{f^k(a)}{k!}(x-a)^k + r_n(x)$

where $r_n(x) = \frac{f^{n+1}(c)}{(n+1)!}(x-a)^{n+1}$

where c is between x and a

Note: If the series alternates then the remainder is < the next term and
La Grange is NOT needed

Famous Maclaurin Series

- sin x = $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$
- cos x = $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$
- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$
- $\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots$
- ln (1+x) = $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots$ only if $-1 < x < 1$
- $\frac{1}{(1-x)} = 1 + x + x^2 + x^3 + x^4 + \ldots$ only if $-1 < x < 1$
- $\frac{1}{(1+x)} = 1 - x + x^2 - x^3 + x^4 - \ldots$ only if $-1 < x < 1$

You need to know everything in this book backwards and forwards.
You need to practice problems until you can do them in your sleep.

Notes: